# Theory of Collective Dynamics in Multi-Agent Complex Systems

Neil F. Johnson<sup>1,3</sup>, Sehyo C. Choe<sup>1</sup>, Sean Gourley<sup>1</sup>, Timothy Jarrett<sup>1</sup>, and Pak Ming Hui<sup>2</sup>

- <sup>1</sup> Physics Department, Clarendon Laboratory, Oxford University, U.K.
- <sup>2</sup> Department of Physics, Chinese University of Hong Kong, Hong Kong
- <sup>3</sup> Corresponding author: n.johnson@physics.ox.ac.uk

Abstract. We discuss a crowd-based theory for describing the collective behavior in Complex Systems comprising multi-agent populations competing for a limited resource. These systems – whose binary versions we refer to as B-A-R (Binary Agent Resource) systems – have a dynamical evolution which is determined by the aggregate action of the heterogeneous, adaptive agent population. Accounting for the strong correlations between agents' strategies, yields an accurate analytic description of the system's dynamics.

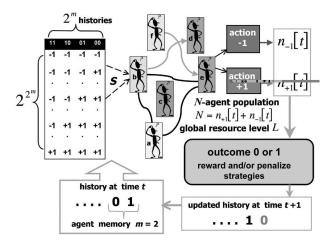
## 1 Introduction

Complex Systems – together with their dynamical behavior known as Complexity – are thought to pervade much of the natural, informational, sociological, and economic world [1,2]. Complex Systems are probably best defined in terms of a list of common features which distinguish them from 'simple' systems, and from systems which are just 'complicated' as opposed to being complex. A list of Complex System 'stylized facts' should include: feedback and adaptation at the macroscopic and/or microscopic level, many (but not too many) interacting parts, non-stationarity, evolution, coupling with the environment, and observed dynamics which depend upon the particular realization of the system.

Casti has argued that [1] '.... a decent mathematical formalism to describe and analyze the [so-called] El Farol Problem would go a long way toward the creation of a viable theory of complex, adaptive systems'. The rationale behind this statement is that the El Farol Problem, which was originally proposed by Brian Arthur [3] to demonstrate the essence of Complexity in financial markets involving many interacting agents, incorporates the key features of a Complex System in an everyday setting. Very briefly, the El Farol Problem concerns the collective decision-making of a group of potential bargoers (i.e. agents) who repeatedly try to predict whether they should attend a potentially overcrowded bar on a given night each week. They have no information about the others' predictions. Indeed the only information available to each agent is global, comprising a string of outcomes ('overcrowded' or 'undercrowded') for a limited number of previous occasions. Hence they

end up having to predict the predictions of others. No 'typical' agent exists, since all such typical agents would then make the same decision, hence rendering their common prediction scheme useless. With the exception of Ref. [4], the physics literature has focused on a simplified binary form of the El Farol Problem called the Minority Game (MG) as introduced by Challet and Zhang [5,6].

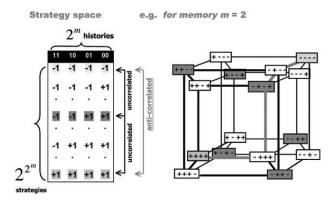
In this paper, we present a theoretical framework for describing a class of Complex Systems comprising competitive multi-agent populations, which we refer to as B-A-R (Binary Agent Resource) systems. The resulting Crowd-Anticrowd theory is not limited to MG-like games, even though we focus on MG-like games in order to demonstrate the accuracy of the approach. The theory is built around the correlations or 'crowding' in strategy-space. Since the theory only makes fairly modest assumptions about a specific game's dynamical behavior, it can describe the dynamics in a wide variety of systems comprising competitive populations [7].



**Fig. 1.** Schematic representation of B-A-R (Binary Agent Resource) system. At timestep t, each agent decides between action -1 and action +1 based on the predictions of the S strategies that he possesses. A total of  $n_{-1}[t]$  agents choose -1, and  $n_{+1}[t]$  choose +1. Agents may be subject to some underlying network structure which may be static or evolving, and ordered or disordered (see Refs. [7,8]). The agents' actions are aggregated, and a global outcome 0 or 1 is assigned. Strategies are rewarded/penalized one virtual point according to whether their predicted action would have been a winning/losing action.

## 2 B-A-R (Binary Agent Resource) Systems

Figure 1 summarizes the generic form of the B-A-R (Binary Agent Resource) system under consideration. At timestep t, each agent (e.g. a bar customer, a commuter, or a market agent) decides whether to enter a game where the choices are action +1 (e.g. attend the bar, take route A, or buy) and action -1 (e.g. go home, take route B, or sell). The global information available to the agents is a common memory of the most recent m outcomes, which are represented as either 0 (e.g. bar attendance below seating capacity L) or 1 (e.g. bar attendance above seating capacity L). Hence this outcome history is represented by a binary bit-string of length m. For general m, there will be  $P=2^m$  possible history bit-strings. These history bit-strings can alternatively be represented in decimal form:  $\mu=\{0,1,...,P-1\}$ . For m=2, for example,  $\mu=0$  corresponds to 00,  $\mu=1$  corresponds to 01 etc. A strategy consists of a predicted action, -1 or +1, for each possible history bit-string. Hence there are  $2^{P=2^m}$  possible strategies.



**Fig. 2.** Strategy Space for m=2, together with some example strategies (left). The strategy space shown is known as the Full Strategy Space (FSS) and contains all possible permutations of the actions -1 and +1 for each history. There are  $2^{2^m}$  strategies in the FSS. The  $2^m$  dimensional hypercube (right) shows all  $2^{2^m}$  strategies in the FSS at its vertices. The shaded strategies form a Reduced Strategy Space RSS. There are  $2 \cdot 2^m = 2P$  strategies in the RSS. The grey shaded line connects two strategies with a Hamming distance separation of 4.

Figure 2 shows the m=2 strategy space from Figure 1. A strategy is a set of instructions to describe what action an agent should take, given any particular history  $\mu$ . The strategy space is the set of strategies from which agents are allocated their strategies. The strategy space shown is known as the Full Strategy Space (FSS) and contains all possible permutations of the actions -1 and +1 for each history. As such there are  $2^{2^m}$  strategies in this space. One can choose a subset of  $2 \cdot 2^m$  strategies, called a Reduced Strategy Space

(RSS), such that any pair within this subset has one of the following two characteristics: (i) Anti-correlated. For example, any two agents using the (m=2) strategies (-1,-1,+1,+1) and (+1,+1,-1,-1) respectively, would take the opposite action irrespective of the sequence of previous outcomes and hence the history. Their net effect on the excess demand  $D[t] = n_{+1}[t] - n_{-1}[t]$  (which is an important quantity in a socio-economic setting such as a financial market) therefore cancels out at each timestep, regardless of the history. Hence they will not contribute to fluctuations in D[t]. (ii) Uncorrelated. For example, any two agents using the strategies (-1,-1,-1,-1) and (-1,-1,+1,+1) respectively, would take the opposite action for two of the four histories, and the same action for the remaining two histories. If the histories occur equally often, the actions of the two agents will be uncorrelated on average. Note that the strategies in the RSS can be labeled by  $R = \{1, 2, ..., 2P = 2.2^m\}$ .

The strategy allocation among agents can be described in terms of a tensor  $\Omega$  describing the distribution of strategies among the N individual agents. This strategy allocation is typically fixed from the beginning of the game, hence acting as a quenched disorder in the system. The rank of the tensor  $\Omega$  is given by the number of strategies S that each agent holds. We note that a single  $\Omega$  'macrostate' corresponds to many possible 'microstates' describing the specific partitions of strategies among the agents. Hence the present Crowd-Anticrowd theory retained at the level of a given  $\Omega$ , describes the set of all games which belong to that same  $\Omega$  macrostate. We also note that although  $\Omega$  is not symmetric, it can be made so since the agents do not distinguish between the order in which the two strategies are picked. Given this, we will henceforth focus on S=2 and consider the symmetrized version of the strategy allocation matrix given by  $\Psi=\frac{1}{2}(\Omega+\Omega^T)$ .

## 3 Crowd-Anticrowd Formalism

Consider an arbitrary timestep t during a run of the game. We will focus on evaluating the 'excess demand'  $D[t] \equiv D[\underline{S}[t], \mu[t]] = n_{+1}[t] - n_{-1}[t]$ , although any other function of  $n_{+1}[t]$  and  $n_{-1}[t]$  can be evaluated in a similar way. Here  $\underline{S}[t]$  is the 2P-dimensional score-vector whose R'th component is the virtual point score for strategy R. [Strategies gain/lose one virtual point at each timestep, according to whether their predicted action would have been a winning/losing action]. The current history is  $\mu[t]$ . The standard deviation of D[t] for this given run, corresponds to a time-average for a given realization of  $\Psi$  and a given set of initial conditions. Summing over the RSS, we have:  $D[\underline{S}[t], \mu[t]] = \sum_{R=1}^{2P} a_R^{\mu[t]} n_R^{\underline{S}[t]}$ . The quantity  $a_R^{\mu[t]} = \pm 1$  is the action predicted by strategy R in response to the history bit-string  $\mu$  at time t. The quantity  $n_R^{\underline{S}[t]}$  is the number of agents using strategy R at time t. We use the

notation  $\langle X[t] \rangle_t$  to denote a time-average over the variable X[t] for a given  $\Psi.$  Hence

$$\langle D\left[\underline{S}[t], \mu[t]\right] \rangle_t = \sum_{R=1}^{2P} \left\langle a_R^{\mu[t]} n_R^{\underline{S}[t]} \right\rangle_t = \sum_{R=1}^{2P} \left\langle a_R^{\mu[t]} \right\rangle_t \left\langle n_R^{\underline{S}[t]} \right\rangle_t \tag{1}$$

where we have used the property that  $a_R^{\mu[t]}$  and  $n_R^{\underline{S}[t]}$  are uncorrelated. We now consider the special case in which all histories are visited equally on average: even if this situation does not hold for a specific  $\Psi$ , it may indeed hold once the averaging over  $\Psi$  has also been taken. For example, in the Minority Game all histories are visited equally at small m and a given  $\Psi$ . If we take the additional average over all  $\Psi$ , then the same is also true for large m. Under the property of equal histories:

$$\langle D\left[\underline{S}[t], \mu[t]\right] \rangle_{t} = \sum_{R=1}^{2P} \left(\frac{1}{P} \sum_{\mu=0}^{P-1} a_{R}^{\mu[t]}\right) \left\langle n_{R}^{\underline{S}[t]} \right\rangle_{t}$$

$$= \sum_{R=1}^{P} \left(\frac{1}{P} \sum_{\mu=0}^{P-1} a_{R}^{\mu[t]} + a_{\overline{R}}^{\mu[t]}\right) \left\langle n_{R}^{\underline{S}[t]} \right\rangle_{t} = \sum_{R=1}^{P} 0. \left\langle n_{R}^{\underline{S}[t]} \right\rangle_{t}$$

$$= 0$$

$$(2)$$

where we have used the exact result that  $a_R^{\mu[t]} = -a_{\overline{R}}^{\mu[t]}$  for all  $\mu[t]$ , and the approximation  $\left\langle n_R^{\underline{S}[t]} \right\rangle_t = \left\langle n_{\overline{R}}^{\underline{S}[t]} \right\rangle_t$ . This approximation is reasonable for a competitive game since there is typically no a priori best strategy: if the strategies are distributed fairly evenly among the agents, this then implies that the average number playing each strategy is approximately equal and hence  $\left\langle n_R^{\underline{S}[t]} \right\rangle_t = \left\langle n_{\overline{R}}^{\underline{S}[t]} \right\rangle_t$ . In the event that all histories are not equally visited over time, even after averaging over all  $\Psi$ , it may still happen that the system's dynamics is restricted to equal visits to some subset of histories. In this case one can then carry out the averaging in Equation (2) over this subspace of histories. More generally, the averagings in this formalism can be carried out with appropriate frequency weightings for each history. In fact, any non-ergodic dynamics can be incorporated if one knows the appropriate history path [8].

The variance of the excess demand D[t] is given by

$$\sigma_{\Psi}^{2} = \left\langle D\left[\underline{S}[t], \mu[t]\right]^{2} \right\rangle_{t} - \left\langle D\left[\underline{S}[t], \mu[t]\right] \right\rangle_{t}^{2} . \tag{3}$$

For simplicity, we will here assume the game output is unbiased and hence  $\langle D\left[\underline{S}[t],\mu[t]\right]\rangle_t=0$ . Hence

$$\sigma_{\Psi}^{2} = \left\langle D\left[\underline{S}[t], \mu[t]\right]^{2} \right\rangle_{t} = \sum_{R, R'=1}^{2P} \left\langle a_{R}^{\mu[t]} n_{R}^{\underline{S}[t]} a_{R'}^{\mu[t]} n_{R'}^{\underline{S}[t]} \right\rangle_{t}. \tag{4}$$

Using the identities  $\underline{a_R}.\underline{a_{R'}} = P$  (fully correlated),  $\underline{a_R}.\underline{a_{R'}} = -P$  (fully anti-correlated), and  $\underline{a_R}.\underline{a_{R'}} = 0$  (fully uncorrelated) where  $\underline{a_R}$  is a vector of dimension P with components  $a_R^{\mu[t]}$  for  $\mu[t] = 1, 2, ..., P$ , we obtain

$$\sigma_{\Psi}^{2} = \sum_{R=1}^{2P} \left\langle \left( n_{R}^{\underline{S}[t]} \right)^{2} - n_{R}^{\underline{S}[t]} n_{\overline{R}}^{\underline{S}[t]} \right\rangle_{t} + \sum_{R \neq R' \neq \overline{R}}^{2P} \left\langle a_{R}^{\mu[t]} a_{R'}^{\mu[t]} \right\rangle_{t} \left\langle n_{\overline{R}}^{\underline{S}[t]} n_{R'}^{\underline{S}[t]} \right\rangle_{t}$$

$$= \sum_{R=1}^{2P} \left\langle \left( n_{R}^{\underline{S}[t]} \right)^{2} - n_{R}^{\underline{S}[t]} n_{\overline{R}}^{\underline{S}[t]} \right\rangle_{t}. \tag{5}$$

The sum over 2P terms can be written equivalently as a sum over P terms,

$$\sigma_{\Psi}^{2} = \sum_{R=1}^{P} \left\langle \left( n_{R}^{\underline{S}[t]} \right)^{2} - n_{R}^{\underline{S}[t]} n_{\overline{R}}^{\underline{S}[t]} + \left( n_{\overline{R}}^{\underline{S}[t]} \right)^{2} - n_{\overline{R}}^{\underline{S}[t]} n_{\overline{R}}^{\underline{S}[t]} \right\rangle_{t}$$

$$= \sum_{R=1}^{P} \left\langle \left( n_{R}^{\underline{S}[t]} - n_{\overline{R}}^{\underline{S}[t]} \right)^{2} \right\rangle_{t} \equiv \left\langle \sum_{R=1}^{P} \left( n_{R}^{\underline{S}[t]} - n_{\overline{R}}^{\underline{S}[t]} \right)^{2} \right\rangle_{t}.$$

The values of  $n_R^{\underline{S}[t]}$  and  $n_{\overline{R}}^{\underline{S}[t]}$  for each R will depend on the precise form of  $\Psi$ . The ensemble-average over all possible realizations of the strategy allocation matrix  $\Psi$  is denoted by  $\langle \ldots \rangle_{\Psi}$ . Using the notation  $\langle \sigma_{\Psi}^2 \rangle_{\Psi} = \sigma^2$ , yields

$$\sigma^2 = \left\langle \left\langle \sum_{R=1}^P \left( n_R^{\underline{S}[t]} - n_{\overline{R}}^{\underline{S}[t]} \right)^2 \right\rangle_t \right\rangle_{\underline{\Psi}} . \tag{6}$$

It is straightforward to obtain analogous expressions for the variances in  $n_{+1}[t]$  and  $n_{-1}[t]$ .

Equation (6) provides us with an expression for the time-averaged fluctuations. Some form of approximation must be introduced in order to reduce Equation (6) to explicit analytic expressions. It turns out that Equation (6) can be manipulated in a variety of ways, depending on the level of approximation that one is prepared to make. The precise form of any resulting analytic expression will depend on the details of the approximations made. Adopting one such approach which is well-suited to the low m regime, we start by relabelling the strategies. Specifically, the sum in Equation (6) is re-written to be over a virtual-point ranking K as opposed to R. Consider the variation in points for a given strategy, as a function of time for a given realization of  $\Psi$ . The ranking (i.e. label) of a given strategy in terms of virtual-points score will typically change in time since the individual strategies have a variation in virtual-points which also varies in time. For the Minority Game, this variation is quite rapid in the low m regime of interest, since there are many more agents than available strategies – hence any strategy emerging as the instantaneously highest-scoring, will immediately get played by many agents and therefore be likely to lose on the next time-step. More general games involving competition within a multi-agent population, will typically generate

a similar ecology of strategy-scores with no all-time winner. This implies that the specific identity of the 'K'th highest-scoring strategy' changes frequently in time. It also implies that  $n_R^{\underline{S}[t]}$  varies considerably in time. Therefore in order to proceed, we shift the focus onto the time-evolution of the highest-scoring strategy, second highest-scoring strategy etc. This should have a much smoother time-evolution than the time-evolution for a given strategy. In the case that the strategies all start off with zero points, the anticorrelated strategies appear as the mirror-image, i.e.  $S_K[t] = -S_{\overline{K}}[t]$ . The label K is used to denote the rank in terms of strategy score, i.e. K=1 is the highest scoring strategy position, K=2 is the second highest-scoring strategy position etc. with

$$S_{K=1} > S_{K=2} > S_{K=3} > S_{K=4} > \dots$$
 (7)

assuming no strategy-ties. Given that  $S_R = -S_{\overline{R}}$  (i.e. all strategy scores start off at zero), then we know that  $S_K = -S_{\overline{K}}$ . Equation (6) can hence be rewritten exactly as

$$\sigma^2 = \left\langle \left\langle \sum_{K=1}^P \left( n_K^{\underline{S}[t]} - n_{\overline{K}}^{\underline{S}[t]} \right)^2 \right\rangle_t \right\rangle_{\Psi}. \tag{8}$$

Since in the systems of interest the agents are typically playing their highest-scoring strategies, then the relevant quantity in determining how many agents will instantanously play a given strategy, is a knowledge of its relative ranking – not the actual value of its virtual points score. This suggests that the quantities  $n_K^{\underline{S}[t]}$  and  $n_K^{\underline{S}[t]}$  will fluctuate relatively little in time, and that we should now develop the problem in terms of time-averaged values. We can rewrite the number of agents playing the strategy in position K at any timestep t, in terms of some constant value  $n_K$  plus a fluctuating term  $n_K^{\underline{S}[t]} = n_K + \varepsilon_K[t]$ . We assume that one can choose a suitable constant  $n_K$  such that the fluctuation  $\varepsilon_K[t]$  represents a small noise term. Hence,

$$\sigma^{2} = \left\langle \sum_{K=1}^{P} \left\langle \left[ n_{K} + \varepsilon_{K}[t] - n_{\overline{K}} - \varepsilon_{\overline{K}}[t] \right]^{2} \right\rangle_{t} \right\rangle_{\Psi}$$

$$\approx \left\langle \sum_{K=1}^{P} \left\langle \left[ n_{K} - n_{\overline{K}} \right]^{2} \right\rangle_{t} \right\rangle_{\Psi} = \left\langle \sum_{K=1}^{P} \left[ n_{K} - n_{\overline{K}} \right]^{2} \right\rangle_{\Psi},$$

$$(9)$$

assuming the noise terms have averaged out to be small. The averaging over  $\Psi$  can now be taken inside the sum. Each term can then be rewritten exactly using the joint probability distribution for  $n_K$  and  $n_{\overline{K}}$ , which we shall call  $P(n_K, n_{\overline{K}})$ . Hence

$$\sigma^{2} = \sum_{K=1}^{P} \left\langle \left[ n_{K} - n_{\overline{K}} \right]^{2} \right\rangle_{\Psi} = \sum_{K=1}^{P} \sum_{n_{K}=0}^{N} \sum_{n_{\overline{K}}=0}^{N} \left[ n_{K} - n_{\overline{K}} \right]^{2} P(n_{K}, n_{\overline{K}}). (10)$$

We now look at Equation (10) in the limiting case where the averaging over the quenched disorder matrix is dominated by matrices  $\Psi$  which are nearly flat. This will be a good approximation in the 'crowded' limit of small m in which there are many more agents than available strategies, since the standard deviation of an element in  $\Psi$  (i.e. the standard deviation in bin-size) is then much smaller than the mean bin-size. The probability distribution  $P(n_K, n_{\overline{K}})$  will then be sharply peaked around the  $n_K$  and  $n_{\overline{K}}$  values given by the mean values for a flat quenched-disorder matrix  $\Psi$ . We label these mean values as  $\overline{n_K}$  and  $\overline{n_{\overline{K}}}$ . Hence  $P(n_K, n_{\overline{K}}) = \delta_{n_K, \overline{n_K}} \delta_{n_{\overline{K}}, \overline{n_K}}$  and so

$$\sigma^2 = \sum_{K=1}^{P} \left[ \overline{n_K} - \overline{n_{\overline{K}}} \right]^2. \tag{11}$$

There is a very simple interpretation of Equation (11). It represents the sum of the variances for each Crowd-Anticrowd pair. For a given strategy K there is an anticorrelated strategy  $\overline{K}$ . The  $\overline{n_K}$  agents using strategy K are doing the opposite of the  $\overline{n_K}$  agents using strategy  $\overline{K}$  irrespective of the history bit-string. Hence the effective group-size for each Crowd-Anticrowd pair is  $n_K^{eff} = \overline{n_K} - \overline{n_K}$ : this represents the net step-size d of the Crowd-Anticrowd pair in a random-walk contribution to the total variance. Hence, the net contribution by this Crowd-Anticrowd pair to the variance is given by

$$[\sigma^2]_{K\overline{K}} = 4pqd^2 = 4.\frac{1}{2}.\frac{1}{2}[n_K^{eff}]^2 = \left[\overline{n_K} - \overline{n_{\overline{K}}}\right]^2$$
 (12)

where p=q=1/2 for a random walk. Since all the strong correlations have been included (i.e. anti-correlations) it can therefore be assumed that the separate Crowd-Anticrowd pairs execute random walks which are *uncorrelated* with respect to each other. [Recall the properties of the RSS - all the remaining strategies are uncorrelated.] Hence the total variance is given by the sum of the individual variances,

$$\sigma^2 = \sum_{K=1}^{P} \left[\sigma^2\right]_{K\overline{K}} = \sum_{K=1}^{P} \left[\overline{n_K} - \overline{n_{\overline{K}}}\right]^2, \tag{13}$$

which corresponds exactly to Equation (11). If strategy-ties occur frequently, then one has to be more careful about evaluating  $\overline{n_K}$  since its value may be affected by the tie-breaking rule. We will show elsewhere that this becomes quite important in the case of very small m in the presence of network connections [8] - this is because very small m naturally leads to crowding in strategy space and hence mean-reverting virtual scores for strategies. This mean-reversion is amplified further by the presence of network connections which increases the crowding, thereby increasing the chance of strategy ties.

#### 4 Implementation of Crowd-Anticrowd Theory

Here we evalute the Crowd-Anticrowd expressions, in the important limiting case of small m. Since there are many more agents than available strategies,

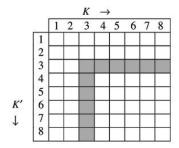


Fig. 3. Schematic representation of the strategy allocation matrix  $\Psi$  with m=2 and S=2, in the RSS. The strategies are ranked according to strategy score, and are labelled by the rank K. In the limit that  $\Psi$  is essentially flat, then the number of agents playing the K'th highest-scoring strategy, is just proportional to the number of shaded bins at that K.

crowding effects will be important. Each element of  $\Psi$  has a mean of  $N/(2P)^S$  agents per 'bin'. In the case of small m and hence densely-filled  $\Psi$ , the fluctuations in the number of agents per bin will be small compared to this mean value – hence the matrix  $\Psi$  looks uniform or 'flat' in terms of the occupation numbers in each bin. Figure 3 provides a schematic representation of  $\Psi$  with m=2, S=2, in the RSS. If the matrix  $\Psi$  is indeed flat, then any reordering due to changes in the strategy ranking has no effect on the form of the matrix. Therefore the number of agents playing the K'th highest-scoring strategy, will always be proportional to the number of shaded bins at that K (see Fig. 3 for K=3). For general m and S, one finds

$$\overline{n_K} = \frac{N}{(2P)^S} [S(2P - K)^{S-1} + \frac{S(S-1)}{2} (2P - K)^{S-2} + \dots + 1]$$

$$= \frac{N}{(2P)^S} \sum_{r=0}^{S-1} \frac{S!}{(S-r)!r!} [2P - K]^r$$

$$= \frac{N}{(2P)^S} ([2P - K + 1]^S - [2P - K]^S)$$

$$= N. \left( \left[ 1 - \frac{(K-1)}{2P} \right]^S - \left[ 1 - \frac{K}{2P} \right]^S \right),$$
(14)

with  $P \equiv 2^m$ . In the case where each agent holds two strategies, S = 2,  $\overline{n_K}$  can be simplified to

$$\overline{n_K} = N. \left( \left[ 1 - \frac{(K-1)}{2P} \right]^2 - \left[ 1 - \frac{K}{2P} \right]^2 \right) = \frac{(2^{m+2} - 2K + 1)}{2^{2(m+1)}} N.$$
 (15)

Hence

$$\sigma^{2} = \sum_{K=1}^{P} \left[ \frac{(2^{m+2} - 2K + 1)}{2^{2(m+1)}} N - \frac{(2K - 1)}{2^{2(m+1)}} N \right]^{2}$$

$$= \frac{N^{2}}{2^{2(2m+1)}} \sum_{K=1}^{P} [2^{m+1} - 2K + 1]^{2} = \frac{N^{2}}{3 \cdot 2^{m}} (1 - 2^{-2(m+1)}) .$$
(16)

This derivation has assumed that there are no strategy ties – more precisely, we have assumed that the game rules governing strategy ties do not upset the identical forms of the rankings in terms of highest virtual points and popularity. Hence we have overestimated the size of the Crowds using high-ranking strategies, and underestimated the size of the Anticrowds using low-ranking strategies. Therefore the analytic form for  $\sigma$  will overestimate the numerical value, as is indeed seen in Figure 4. Notwithstanding this overestimation, there is remarkably good agreement between the numerical results and our analytic theory. In a similar way to the above calculation, the Crowd-Anticrowd theory can be extended to deal with the important complementary regimes of (i) non-flat quenched disorder matrix  $\Psi$ , at small m, and (ii) non-flat quenched disorder matrix  $\Psi$ , at large m. As shown in Figure 4, the agreement for these regimes is also excellent [2,7].

The Crowd-Anticrowd theory has also been applied successfully to various generalizations of the Minority Game. For example, excellent agreement between the resulting analytic expressions and numerical simulations has been demonstrated for (i) Alloy Minority Game [9], (ii) Thermal Minority Game (TMG) [10,11], (iii) Thermal Alloy Minority Game [12], and (iv) B-A-R systems with an underlying network structure [7].

## 5 Conclusion and Discussions

We have given an overview of the Crowd-Anticrowd theory for competitive multi-agent systems, in particular those based on an underlying binary structure. Explicit analytic expressions can be evaluated at various levels of approximation, yielding very good agreement with numerical simulations. We note that the crucial element of this Crowd-Anticrowd theory – i.e. properly accounting for the dominant inter-agent correlations – is not limited to one specific game. Given its success in describing a number of generalized B-A-R systems, we believe that the Crowd-Anticrowd framework could provide a powerful approach to describing a wide class of Complex Systems which mimic competitive multi-agent games. This would be a welcome development, given the lack of general theoretical concepts in the field of Complex Systems as a whole. It is also pleasing from the point of view of physics methodology, since the basic underlying philosophy of accounting correctly for 'inter-particle' correlations is already known to be successful in more

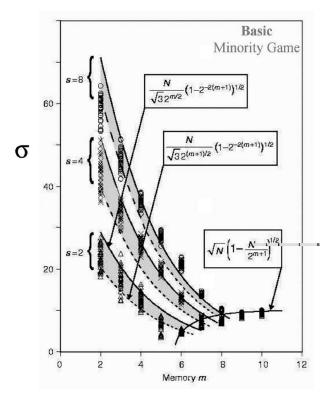


Fig. 4. Crowd-Anticrowd theory vs. numerical simulation results for  $\sigma$  in the Minority Game as a function of memory size m, for N=101 agents, at S=2, 4 and 8. At each S value, analytic forms of  $\sigma$  (i.e. standard deviation in excess demand D[t]) are shown. The numerical values were obtained from different simulation runs (triangles, crosses and circles). Figure adapted from Ref. [2].

conventional areas of many-body physics. This success in turn raises the intriguing possibility that conventional many-body physics might be open to re-interpretation in terms of an appropriate multi-particle 'game': we leave this for future work.

Of course, some properties of Complex Systems cannot be described using time- and configuration-averaged expressions as discussed here. In particular, an observation of a real-world Complex System which is thought to resemble a multi-agent game, may correspond to a *single* run which evolves from a specific initial configuration of agents' strategies. This implies a particular  $\Psi$ , and hence the time-averagings within the Crowd-Anticrowd theory must be carried out for that particular choice of  $\Psi$ . However this problem can still be cast in terms of the Crowd-Anticrowd approach, since the averagings are then just carried out over some sub-set of paths in history space, which is conditional on the path along which the Complex System is already heading.

We have been discussing a Complex System based on multi-agent dynamics, in which both deterministic and stochastic processes co-exist, and are indeed intertwined. Depending on the particular rules of the game, the stochastic element may be associated with any of five areas: (i) disorder associated with the strategy allocation and hence with the heterogeneity in the population, (ii) disorder in an underlying network. Both (i) and (ii) might typically be fixed from the outset (i.e., quenched disorder) hence it is interesting to see the interplay of (i) and (ii) in terms of the overall performance of the system [8]. The extent to which these two 'hard-wired' disorders might then compensate each other, as for example in the Parrondo effect or stochastic resonance, is an interesting question. Such a compensation effect might be engineered, for example, by altering the rules-of-the-game concerning interagent communication on the existing network. Three further possible sources of stochasticity are (iii) tie-breaks in the scores of strategies, (iv) a stochastic rule in order for each agent to pick which strategy to use from the available Sstrategies, as in the Thermal Minority Game, (v) stochasticity in the global resource level L[t] (e.g. bar seating capacity) due to changing external conditions. To a greater or lesser extent, these five stochastic elements will tend to break up any deterministic cycles arising in the game. We refer to Ref. [13] for a discussion of the dynamics of the Minority Game viewed from the perspective of a stochastically-perturbed deterministic system.

### References

- 1. J.L. Casti, Would-be Worlds (Wiley, New York, 1997).
- N.F. Johnson, P. Jefferies, P.M. Hui, Financial Market Complexity (Oxford University Press, 2003).
- 3. B. Arthur, Amer. Econ. Rev. 84, 406 (1994); Science 284, 107 (1999).
- N.F. Johnson, S. Jarvis, R. Jonson, P. Cheung, Y. Kwong and P.M. Hui, Physica A 258, 230 (1998).
- 5. D. Challet and Y.C. Zhang, Physica A 246, 407 (1997).
- See http://www.unifr.ch/econophysics/minority and E. Moro e-print condmat/0402651 at xxx.lanl.gov for the full MG-related literature.
- See N.F. Johnson and P.M. Hui, e-print cond-mat/0306516 at xxx.lanl.gov, for more details.
- S. Gourley, S.C. Choe, P.M. Hui and N.F. Johnson, e-print cond-mat/0401526 at xxx.lanl.gov.
- N.F. Johnson, P.M. Hui, Dafang Zheng, and M. Hart, J. Phys. A: Math. Gen. 32, L427 (1999).
- A. Cavagna, J.P. Garrahan, I. Giardina and D. Sherrington, Phys. Rev. Lett. 83, 4429 (1999).
- M.L. Hart, P. Jefferies, N.F. Johnson and P.M. Hui, Phys. Rev. E 63, 017102 (2001).
- P. Jefferies, M. Hart, N.F. Johnson, and P.M. Hui, J. Phys. A: Math. Gen. 33, L409 (2000).
- 13. P. Jefferies, M.L. Hart and N.F. Johnson, Phys. Rev. E 65, 016105 (2002).